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Diffusion on random hierarchical structures

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Abstract. We study diffusion on a statistically self-similar fractal substrate with random transition rates possessing a hierarchical structure. The master equation is solved analytically by means of a recursion technique and different choices of random hierarchies for the transition rates are considered, leading to algebraic, stretched exponential or exponential-logarithmic behaviour for the moments and the correlation decay. Making use of the central limit theorem, analytical expressions for the fluctuation corrections to lowest order in the relative variances are derived for the first two cases where the behaviour is qualitatively the same as in the systems without fluctuations. The sign of the corrections turns out to depend on the temporal and spatial scale at which fluctuations are externally suppressed; they are negative if the diffusion is taking place at smaller scales, positive otherwise. The relevance to diffusion in a turbulent medium (intermittency corrections) is discussed.

1. Introduction

Anomalous diffusion has recently attracted a great deal of attention, mainly from the point of view of glassy relaxation (for a recent review see Blumen *et al* [1]). Random walks (RW) on percolation clusters [2] and other fractal structures [3], as models for random media, were investigated by many authors [4]. Following the work of Mézard *et al* [5] elucidating the ultrametric character [6] of the equilibrium set of states of the infinite-range spin glass, much effort was directed to the problem of diffusion on ultrametric spaces with a hierarchy of relaxation rates [7-19].

In contrast to RW on such fractal substrates as percolation clusters [2-4] where the variance of the relative displacement, $\langle x^2 \rangle \sim t^{\theta_2}$ with $\theta_2 < 1$, enhanced anomalous diffusion is observed in turbulent media, with the remarkably large exponent $\theta_2 \approx 3$. Wegner, Grossmann and Hoffmann have introduced a model [15-17] where the phase space of the diffusing particle is represented by a Cayley tree with uniform branching ratio z ; the k th level nodes of the tree are associated with intervals of spatial extent μ^k ($\mu > 1$). (The reader is referred to [15] for a detailed discussion.) The nesting property of these intervals leads to the fractal dimension $D_F = \ln z / \ln \mu$ for this space on which the particles are diffusing. Transition rates associated with successive bonds scale by a uniform scale factor s (where s may be less than, equal to or greater than one) leading to a self-similar structure for the eigenvalue spectrum and to algebraic behaviour of the autocorrelation function $\mathcal{P}_0(t) \sim t^{-\nu}$ and of the moments of the displacement from the origin, $\langle x^m \rangle(t) \sim t^{\theta_m}$.

The model's parameter space displays a rich variety of RW with 'phase transitions' [9] between them, as well as 'pseudotransitions' where the dependence of θ_m on the

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parameters, or even the functional form of the moments (algebraic or exponential) themselves, may change abruptly with m (as also observed by Derrida [20]) keeping the other parameters fixed. The exponent ν describing the decay of correlations agrees with the glassy relaxation models where the barrier heights (i.e. the relaxation rates) depend exponentially [7-13] on the ultrametric distance between the states.

The purpose of the present paper is to study the effect of random fluctuations [14, 21-28] in the model parameters on the long time behaviour of the moments and correlations. We replace the homogeneous fractal substrate by a more realistic random fractal structure where the branching ratios and the length scaling factors are allowed to vary randomly from level to level. We moreover allow the transition rates W_k^\pm (where \pm denotes transitions upwards or downwards from level k) to be random. Moments and correlations now have to be averaged not only over different realisations of the RW but also over all possible realisations of the random parameters, which we assume to relax more slowly than the leading transition rates associated with the RW.

To be able to treat the case of arbitrary W_k^\pm a renormalisation-group-like recursion relation (similar in spirit to that considered by Sibani [13]) is developed for the solutions of the master equation (ME) in Laplace space [23], and the eigenvalues obtained analytically in terms of the transition rates. In this way we are able to treat three possible models leading to different distributions for the W_k^\pm :

- (i) The ratios $s_k = W_k^+ / W_{k-1}^+$ are distributed independently from level to level.
- (ii) The $W_k^+ = (\sum_{i=1}^k v_i)^{-\gamma}$, with v_i independently distributed from level to level.
- (iii) $W_k^+ = w_k^k$, and the w_k are distributed independently according to $p(w)$ defined on $(0, 1)$ with a relative variance much smaller than one.

These choices generalise, in a natural way, previously studied models with hierarchically ordered relaxation times [7-11, 15-17]. The first choice (a similar random hierarchy is considered by Schreckenberg [26] and by DeDominicis [27]) reduces on average to relaxation times depending exponentially on the level of the hierarchy and can be used either to model a turbulent medium which has this scaling property, or, say, a glassy structure with energy barriers scaling logarithmically with cluster sizes [24] in the case that the cluster sizes themselves form a (statistically) self-similar hierarchy. In this case the self-similarity of the eigenvalue spectrum, in the ensemble, and therefore the algebraic behaviour of the moments and correlations reported previously [15-17] is preserved (see also [26]). The model affords the extra freedom of choosing the time and length scale at which the system is prepared. Given a reference level L , where the fluctuations are suppressed, e.g. due to external forcing, for $|k - L|$ sufficiently large, the W_k^\pm themselves are distributed according to an approximately log-normal distribution as long as the distribution functions for the s_k satisfy the conditions for the central limit theorem (CLT). Using the CLT we have calculated corrections to the exponents θ_m and ν to first order in the relative variances $\hat{\sigma}^2 \equiv \langle (\delta \ln \cdot)^2 \rangle / \langle \ln \cdot \rangle^2$. The sign of the correction terms depends on whether the diffusion is taking place above (+) or below (-) the level L . The results can be summarised for $s_k < 1$ and $W_{k-1}^+ / W_k^- > 1$ (almost all k) as

$$\theta_m = (\langle \ln \mu \rangle / \langle \ln s^{-1} \rangle) [m \pm \frac{1}{2} m^2 \langle \ln \mu \rangle (\hat{\sigma}_\mu^2 + \hat{\sigma}_s^2)] \tag{1.1}$$

$$\nu = (\langle \ln z \rangle / \langle \ln s^{-1} \rangle) [1 - \frac{1}{2} \langle \ln z \rangle (\hat{\sigma}_z^2 \pm \hat{\sigma}_s^2)] \tag{1.2}$$

where μ_k , the length rescaling factor and z_k , the branching ratio, are independently distributed from level to level, so that the extension and the number of sites at any level also obey approximately log-normal distributions by construction. The result is

valid only up to first order in the variances, so it is assumed that $\langle \ln \mu \rangle m (\hat{\sigma}_\mu^2 + \hat{\sigma}_\nu^2) \ll 1$ and $\langle \ln z \rangle (\hat{\sigma}_z^2 + \sigma_z^2) \ll 1$ in (1.1) and (1.2). We have also calculated the exponent θ_m for a two-delta-function distribution of the ratio W_k^+ / W_{k+1}^- , which is the relevant time rescaling factor in the region $s_k < 1$, $W_k^+ / W_{k+1}^- < 1$, without recourse to the CLT. This allows us to study the corrections due to fluctuations in the case of arbitrary m and variances limited only by the requirement that the qualitative form of the spectrum is preserved.

The second choice (ii) extends models where the relaxation times depend algebraically on the ultrametric distance between states [11] or, say, the cluster sizes determining the energy barriers increase algebraically with the level of the hierarchy [24]. (In the latter case γ is a temperature-dependent exponent.) A similar model has also been studied [27] in the context of the dynamics of the random energy model [29]. With this choice the level where the fluctuations are externally suppressed can be fixed at will by taking

$$(W_k^\pm)^{-1/\gamma} = W_L + \sum_{i=k+1 \text{ or } L+1}^{L \text{ or } k} \text{sgn}(k-L) v_i$$

and under the usual conditions for the distribution of the v_i the W_k^\pm obey approximate log-normal distributions via the CLT. We find stretched exponential behaviour for the moments and correlations in this case as also reported by Rammal [24] and others [7, 11, 25, 27]. It is interesting to note that this is quite independent of the particular choice for the original distribution chosen for the v_i , as long as the leading relaxation rate depends, on average, algebraically on the level of the hierarchy.

The third possibility considered above (iii) is that the W_k^\pm are distributed independently of each other, according to distribution functions that depend on k in such a way as to favour longer relaxation times for higher k . We find that for $p(w)$ with sufficiently small variance and w bounded strictly away from one, only logarithmic corrections are introduced to the power-law behaviour in the region $W_k^+ / W_{k+1}^- > 1$. Allowing $p(w)$ to be finite at $w=1$ leads, within an approximate calculation, to a divergence of all the moments for any t (this corresponds to the result, found in [16, 17] for the pertinent parameter region, that $\langle x^m \rangle(t)$ is infinite for all t or depends exponentially on t). On the other hand, for $p(w) \rightarrow_{w \rightarrow 1} \exp[-(1-w)^{-\alpha}]$, $\alpha > 1$, we find an exponential-logarithmic [30] behaviour for the moments, namely

$$\langle\langle x^m \rangle\rangle_c(t) \sim \exp[C_m(\alpha)(\ln t)^{\alpha/(\alpha-1)}] \tag{1.3}$$

where $C_m(\alpha)$ depends on m and α but not on t and $\langle \dots \rangle_c$ denotes the configurational average. This type of behaviour has been previously reported by Scher and Lax [31] who have considered hopping rates depending on random hopping distances x as $W(x) \sim e^{-\text{constant} \times x}$.

The paper is organised as follows: § 2 presents the full analytical solution of the master equation for an arbitrary realisation of the random parameters, by means of an exact recursion relation. The eigenvalue spectra are obtained in various parameter ranges.

Section 3 deals with the calculation of the quenched averages of the functions $\langle x^m \rangle(t)$ and $\mathcal{P}_0(t)$ and the extraction of the exponents θ_m and ν in case (i) above. Section 4 contains a discussion of the results obtained for θ_m , ν in this case as well as an application of the present model to the case of diffusion in a turbulent medium. Finally, § 5 summarises the results of the computations with the other two models, introducing randomness as discussed above.

2. Derivation and solution of the recursion relation

The space on which the RW is to take place is displayed in figure 1. In the particular case of diffusion in a turbulent medium, the states represent the possible values of the two-particle distance with one of the particles attached to the origin at $k = 0$. Following [15–17], let us define the quantities $q_k^{(l)}(t)$ ($k = 0, \dots, l$) describing the probability of a phase point to be at the k th level at time t without having exceeded the l th level up to this time. The functions $q_k^{(l)}(t)$ turn out to be the quantities needed to obtain the autocorrelation function or the moments of the distance from the origin. Their relationship to the first passage-time problem has been commented upon by Honerkamp and Baumgärtner [18]. Since they are summed quantities, pertaining to clusters of states at level k , with ultrametric distance $\leq l - k$, their dynamical behaviour does not depend on z ; the ME in terms of dimensionless time becomes [16, 17]

$$\dot{q}_k^{(l)}(t) = W_{k-1}^+ q_{k-1}^{(l)}(t) - (W_k^- + W_k^+) q_k^{(l)}(t) + W_{k+1}^- q_{k+1}^{(l)}(t) \tag{2.1}$$

where $q_k^{(l)} = 0$ for $k < 0$ and $k > l$; W_k^\pm are the transition rates from the k th level to the next higher (+) or lower (-) level, respectively. Due to the lower cutoff on the Cayley tree [16], $W_0^- = 0$.

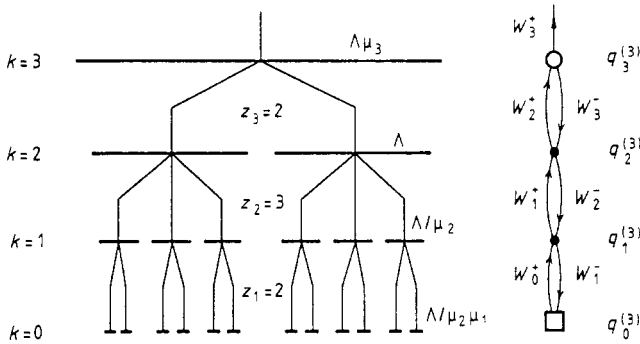


Figure 1. A representation of the lower left corner of a given realisation of the random hierarchical substrate. The levels $k = 0, \dots, 3$ with branching ratios z_k are shown. Each branching point as well as all the endpoints stand for possible states. The horizontal bars at each level k represent the linear extension associated with clusters of states having a common ‘ancestor’ at that level. The extension undergoes a dilation (contraction) by a random scale factor μ_k (μ_k^{-1}) at each successive level above (below) a chosen level (here level 2) at which fluctuations are suppressed due to, say, external forcing. The RHS of the figure illustrates the master equation, (2.1), for the functions $q_k^{(l)}(t)$ (see text); here $l = 3$. Each level k appears as a vertex, with three types of vertices, $k = 0$ (\square), $0 < k < l$ (\bullet), $k = l$ (\circ) to be distinguished from each other.

To derive a recursive solution for the $q_k^{(l)}$, we Laplace transform equation (2.1), and obtain

$$\omega \tilde{q}_k^{(l)}(\omega) - \delta_{k,0} = W_{k-1}^+ \tilde{q}_{k-1}^{(l)}(\omega) - (W_k^- + W_k^+) \tilde{q}_k^{(l)}(\omega) + W_{k+1}^- \tilde{q}_{k+1}^{(l)}(\omega). \tag{2.2}$$

Dividing throughout by $\tilde{q}_k^{(l)}(\omega)$ for $k > 0$ and defining $\phi_k^{(l)}(\omega) \equiv \tilde{q}_k^{(l)}(\omega) / \tilde{q}_{k-1}^{(l)}(\omega)$ as well as $\phi_0^{(l)}(\omega) \equiv \tilde{q}_0^{(l)}$ for $k = 0$, so that

$$\tilde{q}_k^{(l)}(\omega) = \prod_{i=0}^k \phi_i^{(l)}(\omega) \tag{2.3}$$

we obtain the full solution in terms of the recursion relations which follow directly from the Laplace transformed ME:

$$\phi_k^{(l)}(\omega) = \frac{W_{k-1}^+}{\omega + W_k^+ + W_k^- - W_{k+1}^- \phi_{k+1}^{(l)}(\omega)} \tag{2.4}$$

$$\phi_0^{(l)}(\omega) = \frac{1}{\omega + W_0^+ - W_1^- \phi_1^{(l)}(\omega)}. \tag{2.5}$$

In order to study these recursion relations, it turns out to be convenient to introduce the following ratios as auxiliary quantities:

$$W_k^+ / W_{k-1}^+ \equiv s_k \tag{2.6a}$$

$$W_{k-1}^- / W_k^- \equiv r_k \tag{2.6b}$$

$$W_k^+ / W_k^- \equiv u_k \tag{2.6c}$$

with $u_k = r_k s_k$. In what follows the $\{W_k^\pm\}$, $\{s_k\}$, or $\{u_k\}$ may be considered as independently distributed random variables. The choice of $\{s_k\}$ (or $\{u_k\}$) as the independent random variables has a physically interesting consequence, namely, it allows us to pick the level (scale) at which the system is prepared (e.g. stirred) merely by picking

$$W_k^+ = \begin{cases} W_L^+ \prod_{i=k+1}^L s_i^{-1} & k < L \\ W_L^+ \prod_{i=L+1}^k s_i & k > L. \end{cases} \tag{2.7}$$

Clearly, if the s_i are independent random variables, the W_k^+ fluctuate more strongly the further k is from L , due to the increasing number of factors.

We now determine the poles, $\omega_i^{(l)}$, $i = 0, \dots, l$, of the $\tilde{q}_k^{(l)}(\omega)$ for an arbitrary set of transition rates $\{W_k^\pm\}$. These $\omega_i^{(l)}$ are equal in modulus to the eigenvalues $\lambda_i^{(l)}$ of the ME (2.1), which are real and positive, as the following argument [17] proves. The tridiagonal, $(l+1) \times (l+1)$ matrix defined by the ME (2.1) can be brought into symmetric form by a change of variables $q_k^{(l)} = r^{k/2} u_k^{(l)}$ for uniform $r = W_k^+ / W_{k+1}^-$, otherwise arbitrary W_k^+ , thus ensuring that the $l+1$ eigenvalues are real. Furthermore, one may show [17] that this matrix is positive definite. From now on we fix $r_k = r$.

To determine the $\omega_i^{(l)}$, first recall that for $k > l$, $\tilde{q}_k^{(l)} = 0$ and thus $\phi_k^{(l)}(k > l) = 0$. Now note, from equations (2.4), that $\phi_l^{(l)}(\omega)$ has just one pole at

$$\omega_l^{(l)} = -(W_l^+ + W_l^-) \tag{2.8}$$

and no zero, while $\phi_{l-1}^{(l)}(\omega)$ has a zero at $\omega_l^{(l)}$. In fact, it is easy to see from this recursion relation that $\phi_k^{(l)}$ has $(l-k)$ zeros, given by the poles of $\phi_{k+1}^{(l)}$, and $l-k+1$ poles. It then follows from equation (2.3) that the poles of $\tilde{q}_k^{(l)}(\omega)$, all k , are given precisely by the poles of $\phi_0^{(l)}(\omega)$, since the poles of the $\phi_i^{(l)}(\omega)$ successively cancel with the zeros of $\phi_{i-1}^{(l)}(\omega)$, $i = 1, \dots, k$. We therefore need only to determine the $l+1$ poles of $\phi_0^{(l)}(\omega)$.

We now derive a recursion relation to calculate $\phi_0^{(l)}(\omega)$ and determine all its poles. Consider generating a finite, $(l+1)$ -level subset of the Cayley tree as in figure 2 from a site with level-index l , attaching, successively, the lower level branches. Let us associate with this construction, functions $\phi_l^{(l,n)}$, where l indicates the level-index where the construction is initiated, n counts the generations of branches already attached to

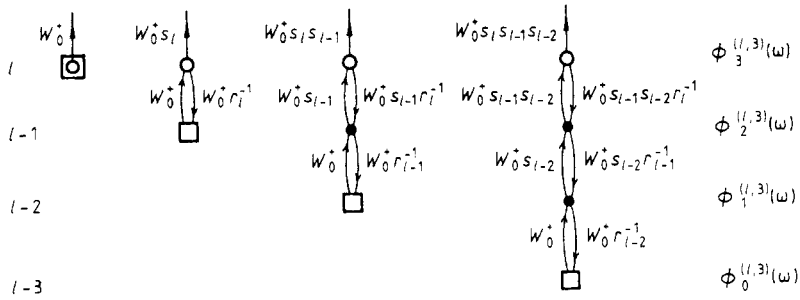


Figure 2. The first steps in the recursion (2.13), starting from a tree consisting of a single vertex (a 1-cluster), extending to a 2-cluster located at l , a 3-cluster, etc, by successively introducing more and more \bullet vertices below l . The recurrence ends after l steps, giving an l cluster which coincides with the original l cluster for which one wants to calculate the $\tilde{q}_k^{(l)}(\omega)$ and whose transition rates are the W_k^* . The $\phi_j^{(l,n)}(\omega)$ of the intermediate steps are indicated. $W_0^+ = 1$ in the text.

l , and j denotes the levels of the $(n + 1)$ cluster, i.e. $j = 0, 1, \dots, n$. Of course, $\phi_j^{(l,n)} \equiv 0$ if $j > n$. Note that $\phi_j^{(l,l)} = \phi_j^{(l)}$. We may show, in particular, that (figure 2)

$$\phi_j^{(l,n)}(\omega) = \phi_{j-1}^{(l,n-1)}(s_{l-n+1}^{-1}\omega). \tag{2.9}$$

$\phi_j^{(l,n)}(\omega)$ and $\phi_{j+1}^{(l,n)}(\omega)$ obey the same relation as equation (2.4), with W_k^* replaced by W_{l-n+k}^* . Putting all this together with equation (2.5), we obtain the recursion relation

$$\phi_0^{(l,n)}(\omega) = \left(\omega + \frac{W_0^+}{1 + W_0^+ u_{l-n+1}^{-1} \phi_0^{(l,n-1)}(s_{l-n+1}^{-1}\omega)} \right)^{-1}. \tag{2.10}$$

Let us briefly comment on the fixed points of recursion (2.10), although we do not use them further. Under repeated iterations of n , $\phi_0^{(l,0)}(\omega)$ is driven to $\phi_0^{(l,n)}$ and the argument ω to $\omega \prod_{j=1}^n s_{l-n+j}$. For any finite ω , this tends to 0 for $s_k < 1$ almost all k , as l (and n) get large. Conversely, for $s_k > 1$, any finite ω will get arbitrarily large. Thus $\omega = 0$ is a stable (unstable) fixed point for $s_k < 1 (> 1)$. Note that the recursion relation in equation (2.10) has the form of a renormalisation group transformation. For uniform u the fixed points for the transformation at $\omega = 0$ are $(\phi^*)^{-1} = (1 - u^{-1}) W_0^+$ and $(\phi^*)^{-1} = 0$, stable, respectively, for $u > 1$ and $u < 1$. In the case of random u_k the transformation at each step depends on the particular realisation of the u_k configuration. For u_k predominantly bigger than one, a finite limit is still approached by $\phi_0^{(l,n)}$ for $n \rightarrow l, l \rightarrow \infty$, whereas for u_k predominantly less than one, $[\phi_0^{(l,n)}]^{-1} \rightarrow 0$ in this limit. This can be seen from the linearised recursion relation, equation (2.15).

Since the $\phi_0^{(l,n)}(\omega)$ are identical to the functions $\tilde{q}_0^{(n)}(\omega)$ with only a relabelling of the levels and the transition rates, they have $n + 1$ poles which lie on the negative real axis for any finite n . They may approach the origin for $l, n \rightarrow \infty$ if $s_k < 1$. Let us now concentrate on this case. Observe from (2.5) that

$$\phi_0^{(l,0)}(\omega) = \frac{1}{\omega + W_0^+} \tag{2.11}$$

which has a pole at $\omega_0^{(l,0)} = -W_0^+$. Substituting in (2.10) gives rise to two poles. For $r > 1$, one gets to lowest order in r^{-1}, s_l (or $(rs_l)^{-1}$ when $rs_l > 1$),

$$\omega_0^{(l,1)} \simeq -W_0^+ \quad \omega_1^{(l,1)} \simeq -W_0^+ s_l.$$

Iteration gives rise, at each successive level, to a new pole at $\omega_0^{(l,n)} = -W_0^+$, while the n poles present at level $n - 1$ are renormalised by s_{l-n+1} appearing in the argument of $\phi_0^{(l,n-1)}$, cf (2.10). This leads to

$$\omega_j^{(l,n)} = -W_0^+ \prod_{i=l-n+1}^{l-n+j} s_i \quad j = 1, \dots, n \quad n = 1, \dots, l. \tag{2.12}$$

For $r < 1$, the poles generated at the first stage of the iteration are

$$\omega_0^{(l,1)} \approx -(1+r^{-1})W_0^+ \quad \omega_1^{(l,1)} \approx -W_0^+ r s_l$$

to lowest order in $r, s_l, r s_l$. Considering $\phi^{(l,n-1)}(s_{l-n+1}\omega)$ at ω near the outermost pole (i.e. that with the largest absolute value) we find for successive $n > 1$, that $\phi^{(l,n)}(\omega)$ has a new pole at $\omega_0^{(l,n)} = -(r^{-1}R_0^{(l,n-1)} + 1)W_0^+$. The residues are $R_0^{(l,0)} = 1, R_0^{(l,1)} \approx r$, while for general n they obey the recursion relation

$$R_0^{(l,n)} = \frac{1 - s_{l-n+1}(r^{-1}R_0^{(l,n-2)} + 1)}{(1 - s_{l-n+1})(r^{-1}R_0^{(l,n-1)} + 1)}$$

for which an attractive fixed point at $r^{1/2} < R_0 < r/s_{l-n+1}$ exists. As in the previous case the poles generated in this way are renormalised at each successive stage by s_{l-n+1} , giving rise to

$$\omega_j^{(l,n)} \approx -r^{-1}W_0^+ \prod_{i=l-n+1}^{l-n+j} s_i \quad j = 1, \dots, n-1 \quad n = 2, \dots, l$$

up to prefactors that do not affect the scaling behaviour. After l iterations one finds $\omega_j^{(l)} = \omega_j^{(l,l)}$ which can be summarised in terms of the transition rates themselves, again up to harmless factors, as

$$\omega_j^{(l)} \approx \begin{cases} W_j^+ & j = 0, \dots, l & r > 1 \\ W_{j+1}^- & j = 0, \dots, l-1 & r < 1. \end{cases} \tag{2.13}$$

The pole $\omega_l^{(l)}$ in the case $r < 1$ turns out to be well separated from the bulk of the poles given in (2.13) and very close to the origin (cf [16, 17]). To determine its position precisely, we consider the recursion relation (2.10) near the origin, putting $W_0^+ \equiv 1$ for simplicity from now on.

Let us make the ansatz, for small ω ,

$$\phi_0^{(l,n)}(\omega) = \{[\phi_0^{(l,n)}(0)]^{-1} + (1 + a_n)\omega\}^{-1}. \tag{2.14}$$

For $s_k > 1$, we are forced to take $a_n \equiv 0$, since any finite ω is driven to infinity under repeated iterations. For $s_k < 1$, the a_n term remains finite and renormalises the pole. Then, the pole of $\phi_0^{(l,n)}(\omega)$ will be given by the expression $-[\phi_0^{(l,n)}(0)]^{-1}(1 + a_n)^{-1}$. We get, from (2.10), recursion relations

$$\phi_0^{(l,n)}(0) = u_{l-n+1}^{-1} \phi_0^{(l,n-1)}(0) + 1 \tag{2.15}$$

$$a_n = u_{l-n+1}^{-1} s_{l-n+1}^{-1} \left(\frac{\phi_0^{(l,n-1)}(0)}{\phi_0^{(l,n)}(0)} \right)^2 (1 + a_{n-1}). \tag{2.16}$$

From (2.15) we obtain

$$\tilde{q}_0^{(l)}(0) \equiv \phi_0^{(l,l)}(0) = 1 + \sum_{k=1}^l \prod_{n=k}^l u_{l-n+1}^{-1}. \tag{2.17}$$

Let us turn briefly to the case of uniform r, s . Performing the sum in (2.17)

$$\tilde{q}_0^{(l)}(0)_{\text{unif}} = \frac{1 - u^{-(l+1)}}{1 - u^{-1}}. \tag{2.18}$$

We distinguish two markedly different regimes:

$$[\tilde{q}_0^{(l)}]^{-1} \sim \begin{cases} (1 - u)u^l & u < 1 \\ 1 - u^{-1} & u > 1. \end{cases} \tag{2.19}$$

Substituting in (2.16) we find, again for uniform r, s , with $s < 1$

$$a_l = \begin{cases} -r(r^l - 1)(1 - r)^{-1} & u < 1 \\ -[(us)^{-l} - 1](us - 1)^{-1} & u > 1. \end{cases} \tag{2.20}$$

The results for the additional pole of $\phi_0^{(l)}$ obtained in this way are summarised in table 1 for the uniform case, and the relevant parameter ranges indicated in figure 3. They coincide with those given in [16, 17].

Analogous results are obtained for the case of random s_k . If the series for $\tilde{q}_0^{(l)}(0)$ and a_l are approximated by their largest terms in each of the different ranges, where it will be assumed that the u_k typically are < 1 (or > 1) and $r \geq 1$ almost all k , etc, we have

$$\begin{aligned} -\omega_l^{(l)} &\approx \prod_{k=1}^l u_k && \text{B, C} \\ &\approx \prod_{k=1}^l s_k && \text{A}_2 \\ &\gg \prod_{k=1}^l s_k && \text{A}_{1,2} \\ &\approx O(1) && \text{A}_{1,1}, \text{D.} \end{aligned} \tag{2.21}$$

Let us recall that the rest of the eigenvalue spectrum is directly given by the transition rates themselves. It is useful to index the eigenvalues in increasing order. Then the bulk of the eigenvalues is given, for $s_k < 1$ (almost all k), to leading order in r^{-1}

$$\lambda_k^{(l)} \approx W_{l-k}^+ + W_{l-k+1}^- \quad k = 1, \dots, l \tag{2.22}$$

Table 1. The extra eigenvalue $\tilde{\lambda}_0^{(l)}$ in the various regions. In the regions B and C, $\tilde{\lambda}_0^{(l)}$ is smaller than and well separated from the bulk of the eigenvalues. In region A₂ it merely joins, as the smallest member, the sequence of eigenvalues in the bulk. In A_{1,2}, since $(rs) > 1$, it is clearly much larger than the smallest eigenvalue in this sequence. In regions A_{1,1} and D it is a constant O(1) with no dependence on l . Here the smallest eigenvalue is s^l and stems from the bulk.

$s < 1$				
$rs < 1$	$rs > 1$		$s > 1$	
$r < 1$ (B)	$r > 1$ (A ₂)	$rs^2 < 1$ (A _{1,2})	$rs^2 > 1$ (A _{1,1})	$rs < 1$ (C) $rs > 1$ (D)
$(1 - r)(1 - rs)(rs)^l$	$(1 - rs)(r - 1)s^l/r$	$[1 - (rs)^{-1}](rs)^l s^l$	$[1 - (rs^2)^{-1}] \times [1 - (rs)^{-1}]$	$(1 - rs)(rs)^l$ $1 - (rs)^{-1}$

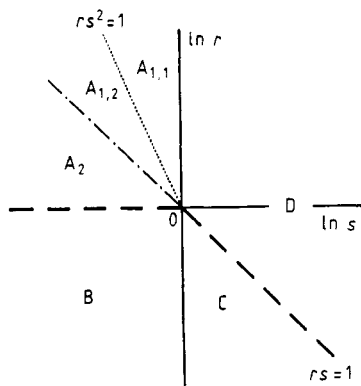


Figure 3. The notation for the various ranges in the parameter space $(\ln s, \ln r)$ that have different eigenvalue spectra (cf [16]). Different diffusion laws and different correlation decays hold in A, B, C, D, and, of course, on the corresponding boundaries which are not considered here: these boundaries are smeared out due to the fluctuations of the r_k, s_k .

from (2.13). From (2.21) we get (again for $s_k < 1$),

$$\lambda_0^{(l)} \approx \begin{cases} \prod_{k=1}^l W_k^+ / W_k^- & \text{in B, C} \\ \prod_{k=1}^l r_k^{-1} W_k^+ / W_k^- = W_l^+ & \text{in A}_2 \\ W_l^+ & \text{in A}_1. \end{cases} \tag{2.23}$$

Having thus obtained all the eigenvalues ((2.22), (2.23)) directly in terms of the transition rates themselves, we now have a choice as to which variables to consider as independently distributed. In the following and in § 3 we will consider the case where the $\{s_k\}$ or the $\{u_k\}$ are taken as independent random variables. In § 5 the cases $W_k^+ = (\sum_i^k v_i)^{-\gamma}$ and $W_k^+ = w_k^k$ will be considered, with v_i and w_k being independently distributed.

Finally, the function $\tilde{q}_i^{(l)}(\omega)$ is

$$\tilde{q}_i^{(l)}(\omega) = \prod_{k=0}^l \phi_k^{(l)}(\omega) \approx \frac{(1+a_l)^{-1} \prod_{k=0}^{l-1} W_k^+}{\prod_{k=0}^l (\omega + \lambda_k^{(l)})}. \tag{2.24}$$

The inverse Laplace transform yields

$$q_i^{(l)}(t) = \sum_{i=0}^l c_i^{(l)} e^{-\lambda_i^{(l)} t} \tag{2.25}$$

$$c_i^{(l)} = \prod_{i=1}^l s_i^{-1} \lambda_i^{(l)} \prod_{\substack{k=0 \\ k \neq i}}^l \frac{\lambda_k^{(l)}}{\lambda_k^{(l)} - \lambda_i^{(l)}}. \tag{2.26}$$

We have used in (2.25) the explicit forms of W_k^+ , $(1+a_l)$ and $\lambda_k^{(l)}$, and the fact that for $u_k < 1$ one has

$$\prod_{k=0}^{l-1} W_k^+ \sim \prod_{k=1}^l \lambda_k^{(l)} r^l$$

whereas for $u_k > 1$

$$\prod_{k=0}^{l-1} W_k^+ \sim \prod_{k=0}^l \lambda_k^{(l)} \prod_{k=1}^l s_k^{-1}.$$

As in [16, 17] $q_i^{(l)}(t)$ may be written

$$q_i^{(l)}(t) = \begin{cases} g_i(\lambda_0^{(l)} t) & \text{in A} \\ c \prod_i^l r_i \exp\left(-\prod_i^l u_i t\right) G_i(\lambda_1^{(l)} t) & \text{in B.} \end{cases} \tag{2.27}$$

$$q_i^{(l)}(t) = \begin{cases} g_i(\lambda_0^{(l)} t) & \text{in A} \\ c \prod_i^l r_i \exp\left(-\prod_i^l u_i t\right) G_i(\lambda_1^{(l)} t) & \text{in B.} \end{cases} \tag{2.28}$$

The c is an l -independent constant. The function $g_i(\lambda_0^{(l)} t)$ may be obtained by straightforward substitution of (2.26) in (2.25). In region B making the approximation $\lambda_0^{(l)} \ll \lambda_i^{(l)}$ we obtain

$$G_l(\tau) = 1 - \sum_{i=1}^l \prod_{\substack{j=1 \\ j \neq i}}^l \frac{\lambda_j^{(l)}}{\lambda_j^{(l)} - \lambda_i^{(l)}} \exp[-\tau(\lambda_i^{(l)} / \lambda_1^{(l)})]. \tag{2.29}$$

With the choice $W_k^+ = W_L \prod_{i=k+1}^L s_i^{-1}$, we have in region A₁, where $\lambda_0^{(l)} = W_l^+(1 + 1/u_l)$,

$$g_i(\lambda_0^{(l)} t) = W_L \prod_{k=1}^L s_k^{-1} \sum_{i=0}^l \frac{(-1)^i \prod_{k=l-i+1}^l s_k^{l-k} \exp(-c_l W_L \prod_{k=l+1}^L s_k^{-1} t / \prod_{k=l-i}^l s_k)}{\prod_{j=i+1}^l (1 - b_{ij} \prod_{k=l-j+1}^{l-i} s_k) \prod_{j=0}^{i-1} (b_{ij} - \prod_{k=l-i+1}^{l-j} s_k)} \tag{2.30}$$

$$c_l = 1 + 1/u_l \quad b_{ij} = (1 + u_i^{-1})(1 + u_j^{-1})^{-1}.$$

The case A₂, where $\lambda_0^{(l)} = W_l^+$, is obtained simply by dropping the factor c_l from the exponent above. In region B,

$$G_l(\lambda_1^{(l)} t) = 1 - W_L \prod_{k=1}^L s_k^{-1} \sum_{i=0}^l \frac{(-1)^i \prod_{k=l-i+1}^l s_k^{l-k+1} \exp(-W_L \prod_{k=l}^L s_k^{-1} t / \prod_{k=l-i}^l s_k)}{\prod_{j=i+1}^l (1 - b_{ij} \prod_{k=l-j+1}^{l-i} s_k) \prod_{j=0}^{i-1} (b_{ij} - \prod_{k=l-i+1}^{l-j} s_k)}. \tag{2.31}$$

These expressions reduce to $g_s(\tau)$ and $G_s(\tau)$ of [16, 17] when the s_k are taken to be uniform and the limits $l \rightarrow \infty$ taken in the sum over i (the terms with $i > l$ hardly contribute because of the exponential factor) as well as in the product over $j = i + 1, \dots, l$, which is essentially $O(1)$, so that the only non-trivial l dependence remaining is in the exponent. These functions continue to behave in much the same way as in the pure case as long as the s_k have small relative variances: $g(\tau)$ is a strongly peaked function at $\tau \sim O(1)$ and $G(\tau)$ behaves like a step function, being essentially zero for $\tau < O(1)$ and 1 for $\tau > O(1)$. It should be noted from (2.30) and (2.31) that $g(\tau) = dG(\tau)/d\tau$.

3. Moments and autocorrelation functions

A knowledge of the time-dependent probability distribution functions $q_k^{(l)}(t)$ enables us to obtain the long-time behaviour of all the moments of the displacement of the random walker from the origin as well as the autocorrelation function. Allowing for the linear scale factor μ and the transition rates to be independently distributed random variables, and singling out level L as the ‘deterministic’ scale (no fluctuations) at which the system is prepared, with length scale Λ_L , we have for the moments

$$\langle\langle x^m \rangle\rangle_c(t) = \Lambda_L^m \left\langle \sum_{l=0}^L \prod_{k=l+1}^L \mu_k^{-m} (\hat{q}^{(l)}(t) - \hat{q}^{(l-1)}(t)) + \sum_{l=L+1}^{\infty} \prod_{k=L+1}^l \mu_k^m (\hat{q}^{(l)}(t) - \hat{q}^{(l-1)}(t)) \right\rangle_c \tag{3.1}$$

where $\langle \dots \rangle_c$ indicates the configurational average over all random variables.

$$\hat{q}^{(l)}(t) = \sum_{k=0}^l q_k^{(l)}(t) \tag{3.2}$$

is the probability for the diffusing particle to be found somewhere between levels 0 and l , provided it has never exceeded level l . The $\hat{q}^{(l)}(t)$ depend implicitly on the random transition rates through the eigenvalues.

Provided that the μ_k are independently distributed from level to level, performing the μ average yields

$$\begin{aligned} \langle\langle x^m \rangle\rangle_c(t) = & \Lambda_L^m \sum_{l=0}^L \exp[(l-L)U_m^-] \langle \hat{q}^{(l)}(t) - \hat{q}^{(l-1)}(t) \rangle_c \\ & + \Lambda_L^m \sum_{l=L+1}^{\infty} \exp[(l-L)U_m^+] \langle \hat{q}^{(l)}(t) - \hat{q}^{(l-1)}(t) \rangle_c \end{aligned} \tag{3.3}$$

where $U_m^\pm = \pm \ln \langle \mu^{\pm m} \rangle_\mu$. Here $\langle \dots \rangle_\mu$ denotes the average with respect to the μ distribution if it is identical for all levels. In the more general case it means

$$\left\langle \prod_{k=l+1}^L \mu_k^{\pm m} \right\rangle^{1/(L-l)}.$$

Note that U_m^\pm has in general a non-linear dependence on m (this will be made explicit in the later discussion involving the use of the central limit theorem, cf (3.20)).

The function $\hat{q}^{(l)}(t)$ satisfies, via (2.1), the relation

$$\dot{\hat{q}}^{(l)}(t) = -W_l^+ q_l^{(l)}(t) \tag{3.4}$$

where $q_l^{(l)}(t)$ is given explicitly by (2.27) and (2.28).

Case (i). From this point on, till the end of this section, we confine ourselves to the case (i) considered in the introduction, and choose the transition rates W_k^+ to be of the form given in (2.7), with the ratios $s_k = W_k^+ / W_{k-1}^+$ being independently distributed random variables. (Cases (ii) and (iii) will be treated in § 5.)

Let us now concentrate on region B, where the lowest eigenvalue, $\lambda_0^{(l)} \approx \prod_{k=1}^l u_k$, is well separated from the others, i.e. $\lambda_0^{(l)} \ll \lambda_1^{(l)}$. It is easy to show under the condition

$$\mu_k^m u_k < 1 \quad \text{almost all } k \tag{3.5}$$

that the infinite series in (3.3) converges uniformly with respect to t , thus allowing us to interchange the sum over l and the derivative with respect to t . Fixing $L = 0$, we have from (2.7)

$$q_l^{(l)}(t) = (W_l^+)^{-1} \lambda_0^{(l)} \exp(-\lambda_0^{(l)} t) G_l(\lambda_1^{(l)} t)$$

which, with (3.4) and (3.3) yields

$$\langle\langle \dot{x}^m \rangle\rangle_c(t) = \sum_{l=0}^{\infty} \exp(lU_m^+) \langle \lambda_0^{(l-1)} G_{l-1}(\lambda_1^{(l-1)} t) - \lambda_0^{(l)} G_l(\lambda_1^{(l)} t) \rangle_c$$

where the time derivative on the RHS has been taken term by term (we set $\Lambda_0 \equiv 1$). Note that with condition (3.5), each of the two terms in the sum above tends to zero with l , separately. Taking the one-pole approximation to $q_l^{(l)}(t)$, i.e. letting $G(\tau) \sim O(1)$, we get

$$\langle\langle \dot{x}^m \rangle\rangle_c(t) \sim \sum_{l=0}^{\infty} \exp(lU_m^+) \langle \lambda_0^{(l)} \exp(-\lambda_0^{(l)} t) \rangle_c. \tag{3.6}$$

Had we not taken $L = 0$ to begin with, the expression within $\langle \dots \rangle_c$ would have been,

e.g. for $l < L$,

$$W_L \prod_{k=l+1}^L s_k^{-1} r^l \exp\left(-\prod_{k=1}^l r s_k t\right).$$

In physical terms, $L=0$ corresponds to a situation where the ‘impurities’ are being introduced at the same scale as that at which the system is prepared and we consider only the time evolution at scales larger than this initial scale.

We consider the two- δ -function distribution for the u_k

$$p(u_k) = p_a \delta(u_k - u_a) + p_b \delta(u_k - u_b). \tag{3.7}$$

For small enough variance, $\sigma_u^2 = p_a p_b (u_a - u_b)^2$, such that the overall form of the spectrum is preserved and the well separated eigenvalue can still be distinguished ($\lambda_0^{(l)} \ll \lambda_1^{(l)}$), the approximate expression in (3.6) is valid. Performing the configuration average yields

$$\langle \lambda_0^{(l)} \exp(-\lambda_0^{(l)} t) \rangle_c = \sum_{n=0}^l \binom{l}{n} p_a^n p_b^{l-n} u_a^n u_b^{l-n} \exp(-u_a^n u_b^{l-n} t). \tag{3.8}$$

Making a saddle-point approximation to the double sum over n and l and neglecting corrections to the saddle point which introduce logarithmic factors in t , one finds $\langle x^m \rangle_c(t) \sim t^{\theta_m}$, with

$$\theta_m = \frac{U_m^+ + \xi \ln(P_a/\xi) + (1 - \xi) \ln[P_b/(1 - \xi)]}{\ln(u_b^{\xi-1}/u_a^\xi)}. \tag{3.9}$$

The parameter ξ is just the ratio of the saddle point in n and l and is to be determined from

$$\ln u_a \ln(1 - \xi) - \ln u_b \ln \xi = U_m^+ \ln(u_a/u_b) + \ln u_a \ln p_b - \ln u_b \ln p_a. \tag{3.10}$$

For $u_a = u_b$, θ_m reduces to $\theta_m^{(0)} = m \ln \mu / \ln u^{-1}$ as found in [16, 17]. For u_a different from u_b , ξ and therefore θ_m depend non-linearly on m even in the case of uniform μ , for which $U_m^+ = m \ln \mu$. It should be noted that $\theta_0 = 0$, as it should be. Initially θ_m against U_m^+ has a slope $\sim (\ln u^{-1})^{-1}$. For very large m , one sees from (3.10) that $\xi \rightarrow 1$, and $\theta_m \sim (\ln p_a + U_m^+) / \ln u_a^{-1}$ for the choice $u_a > u_b$. Thus for very large m , θ_m is once more linear in U_m^+ , this time with a slope $1/\ln u_a^{-1} > 1/(\ln u^{-1})$, the latter being the slope in the pure case, if we identify $(\ln u^{-1})$ taken with respect to the distribution in (3.7), with the value of $\ln u^{-1}$ in the pure case. We have sketched the behaviour of θ_m with respect to U_m^+ (or m , for uniform μ) in figure 4.

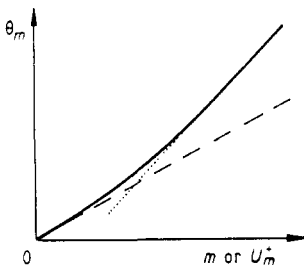


Figure 4. The moment growth exponent θ_m against m or U_m^+ , schematic. The initial slope is determined by $1/(\ln u^{-1})$, the final slope by $u_a (> u_b)$, namely, $1/\ln u_a^{-1}$. The location of the crossover region depends on the variance of the u fluctuations: $m(\ln \mu) \sigma_u^2 \ll O(1)$.

To be able to evaluate the configuration averages in the more general case, we make the observation that if the u_k are independent random variables and the u_k distributions fulfil the conditions of the CLT, then $y = \ln \lambda_0^{(l)} = \sum_{k=1}^l \ln u_k$ will be approximately distributed according to

$$P_l(y) = \frac{1}{\sqrt{2\pi l\sigma_u^2}} \exp\left(-\frac{1}{2} \frac{(y - l\bar{y}_u)^2}{l\sigma_u^2}\right) \tag{3.11}$$

where

$$\begin{aligned} \bar{y}_u &= \langle \ln u \rangle_u \\ \sigma_u^2 &= \langle (\delta \ln u)^2 \rangle_u. \end{aligned} \tag{3.12}$$

Here $\langle \dots \rangle_u$ means the average taken with respect to the u distribution. In this case this depends on the level

$$\langle \dots \rangle_u = \frac{1}{|l-L|} \sum_{k=l+1}^{L \text{ or } L+1} \langle \dots \rangle_{u_k}.$$

In what follows, similar averages will occur for s_k, μ_k, z_k , etc, which should be understood in the same way. We will henceforth drop the index u (s, μ, \dots) where no confusion is likely to arise. Care should be taken, however, in calculating moments using (3.11). It has already been remarked by Mandelbrot [32] that higher moments of an approximately log-normal variable differ increasingly from their true values in as much as the original distributions differ from the Gaussian. In particular it is easy to see that the calculation of $\langle \lambda_0^{(l)} \rangle = \langle e^y \rangle$ using the distribution in (3.11) yields results that are only correct up to $O(\sigma_u^2)$ for a general u distribution. Therefore we restrict ourselves to distributions with sufficiently small variances. Evaluating

$$\langle \lambda_0^{(l)} \exp(-\lambda_0^{(l)} t) \rangle_c = \int_{-\infty}^{\infty} P_l(y) \exp(y - e^y t) dy$$

via a saddle-point approximation gives the condition

$$-\frac{\hat{y}_l - l\bar{y}_u}{l\sigma_u^2} + 1 - t \exp \hat{y}_l = 0 \tag{3.13}$$

where \hat{y}_l is the value of y at the saddle point. Substituting in equation (3.6) gives

$$\langle \langle \dot{x}^m \rangle \rangle_c(t) \sim \sum_{i=0}^{\infty} \exp(iU_m^+) \exp\left(-\frac{1}{2} \frac{(\hat{y}_l - l\bar{y}_u)^2}{l\sigma_u^2} + \hat{y}_l - t \exp(-\hat{y}_l)\right) \tag{3.14}$$

neglecting corrections to the saddle point as well as the normalisation factor, both of which eventually introduce logarithmic corrections in t but which have no effect on the exponent. Once more using the saddle-point approximation to evaluate the sum over l we get the additional condition

$$U_m^+ + \bar{y} \frac{(\hat{y}_l - l\bar{y}_u)}{\bar{l}\sigma_u^2} + \frac{1}{2} \frac{(\hat{y}_l - l\bar{y}_u)^2}{\bar{l}^2\sigma_u^2} = 0 \tag{3.15}$$

for the saddle point \bar{l} , which is a function of t . One realises that the quantity $y_0 = \hat{y}_l / \bar{l}$ satisfies a quadratic equation with \bar{l} -independent coefficients, i.e. does not depend on \bar{l} . One then has

$$y_0 = (\bar{y}_u^2 - 2\sigma_u^2 U_m^+)^{1/2} \tag{3.16}$$

which is also independent of t . Introducing $\hat{y}_l = \bar{I}y_0$ into (3.13) gives

$$\bar{I} = -\frac{\ln t}{y_0} + \frac{\ln[1 - \sigma_u^{-2}(y_0 - \bar{y}_u)]}{y_0} \approx -\frac{\ln t}{y_0}. \tag{3.17}$$

Inserting this into (3.14) we recover, up to logarithmic factors in t , the algebraic form

$$\langle\langle x^m \rangle\rangle_c(t) \sim t^{\theta_m} \tag{3.18}$$

where θ_m turns out to be $\theta_m = (y_0 - \bar{y}_u)/\sigma_u^2$. Substituting from (3.16),

$$\theta_m = -\frac{\bar{y}_u}{\sigma_u^2} [1 - (1 - 2\sigma_u^2 U_m^+/\bar{y}_u^2)^{1/2}]. \tag{3.19}$$

We again see that the dependence of θ_m on m is in general non-linear.

Applying the CLT once more to the μ distribution, we may approximate U_m^\pm to first order in $\sigma_\mu^2 = \langle(\delta \ln \mu)^2\rangle$ by

$$U_m^\pm \approx m \langle \ln \mu \rangle \pm \frac{1}{2} m^2 \sigma_\mu^2. \tag{3.20}$$

Then, expanding (3.19) to first order in the relative variances $\hat{\sigma}^2 \equiv \langle(\delta \ln \cdot)^2\rangle/\langle \ln \cdot \rangle^2$ we find

$$\theta_m = \theta_1^{(0)} [m + \frac{1}{2} m^2 \langle \ln \mu \rangle (\hat{\sigma}_\mu^2 + \hat{\sigma}_u^2)] \tag{3.21}$$

where $\theta_1^{(0)} \equiv \langle \ln \mu \rangle / \langle \ln u^{-1} \rangle$. Thus, for $L = 0$, the correction to θ_m is increasing quadratically with m for $\langle \ln \mu \rangle m (\hat{\sigma}_\mu^2 + \hat{\sigma}_u^2) \ll 1$. Expanding the exact result for θ_m calculated for the two- δ -function distribution (3.7) to first order in the variances, we recover (3.21). Hoffmann, who considered fluctuations in μ [33], also found correction terms proportional to m^2 in lowest order of the μ variance as well as in $\langle \delta \mu_k \delta \mu_{k+1} \rangle$, the nearest-level correlation for the μ fluctuations, for a restricted time range. Setting $\hat{\sigma}_\mu^2 = \hat{\sigma}_u^2 = 0$ in (3.21) we recover the results given in [16, 17] for the corresponding parameter region. Again, for $m = 0$, $\theta_0 = 0$.

Let us now turn to the parameter region A where there is no well separated eigenvalue. From (2.22) and (2.23) we see that in regions A_1 and A_2

$$\lambda_k^{(l)} = W_{l-k}^+ (1 + r^{-1}) \quad k = 1, \dots, l$$

while $\lambda_0^{(l)} = W_l^+$. Notice that the presence or absence of the factor $(1 + r^{-1})$ does not change the asymptotic dependence on l of the eigenvalues $\lambda_k^{(l)}$, apart from introducing time-independent factors in the averages, and we may therefore treat A_1, A_2 on the same footing. For later convenience we choose $\ln W_L = L \langle \ln s \rangle$.

Once more making use of (3.4), with (2.27) and the fact that $g(\tau) = dG(\tau)/d\tau$, we may rewrite (3.3)

$$\begin{aligned} \langle\langle x^m \rangle\rangle_c(t) &= \Lambda_L^m \sum_{l=0}^L \exp[(l-L)U_m^-] \langle G_{l-1}(\lambda_0^{(l-1)}t) - G_l(\lambda_0^{(l)}t) \rangle_c \\ &+ \Lambda_L^m \sum_{l=L+1}^\infty \exp[(l-L)U_m^+] \langle G_{l-1}(\lambda_0^{(l-1)}t) - G_l(\lambda_0^{(l)}t) \rangle_c. \end{aligned} \tag{3.22}$$

Let us consider for now the first sum appearing above. We define as before the variable $y = \sum_{k=l+1}^L \ln s_k$ distributed, under the usual conditions, according to

$$P_{L-l}(y) \sim \exp\{-\frac{1}{2}[y - (L-l)\bar{y}_s]^2 / (L-l)\sigma_s^2\}$$

with

$$\bar{y}_s = \langle \ln s \rangle \quad \sigma_s^2 = \langle (\delta \ln s)^2 \rangle.$$

We note that for sufficiently large l the difference $\Delta_l G \equiv G_{l-1}(\lambda_0^{(l-1)} t) - G_l(\lambda_0^{(l)} t)$ is a very sharply peaked function at $\lambda_0^{(l)} t \sim 1$, falling away to zero everywhere else. Going over from the sum to an integral over l and approximating $\Delta_l G$ by $\delta(\ln t + \ln W_L - y)$, we have up to constant factors

$$\langle\langle x^m \rangle\rangle_c(t) \sim \Lambda_L^m \int_0^L dl \int_{-\infty}^{\infty} dy \delta(\ln t + L\bar{y}_s - y) P_{L-l}(y) \exp[(l-L)U_m^-]. \tag{3.23}$$

Clearly, for $\bar{y}_s < 0$, the integral is non-zero only for $\ln t / (-\bar{y}_s) < L$ (positive values of y are rapidly suppressed by the Gaussian factor). We obtain

$$\langle\langle x^m \rangle\rangle_c(t) \sim \Lambda_L^m \int_0^L dl \exp[(l-L)U_m^-] P_{L-l}(\ln t + L\bar{y}_s) \tag{3.24}$$

where the distribution over y has been effectively converted to a distribution function for l :

$$P_{L-l}(\ln t + L\bar{y}_s) \sim \exp\left(-\frac{1}{2} \frac{(\ln t + l\bar{y}_s)^2}{(L-l)\sigma_s^2}\right).$$

We substitute for l in the denominator its mean value, $\bar{l} = \ln t / (-\bar{y}_s)$, observing that the denominator is already $O(\sigma_s^2)$, in the same spirit as before. The Gaussian integral can now easily be performed. If we once more make the log-normal approximation to the μ distribution (3.20), and choose $\ln \Lambda_L = L(\ln \mu)$ for convenience, we find in analogy to (3.18), (3.21),

$$\langle\langle x^m \rangle\rangle_c(t) \sim t^{\theta_m^{(0)}} (t/T)^{-\frac{1}{2} m \theta_m^{(0)} (\ln \mu) (\hat{\sigma}_\mu^2 + \hat{\sigma}_s^2)} \tag{3.25}$$

up to logarithmic factors in t . Here $\ln T \equiv L|\bar{y}_s|$ and we have defined $\theta_m^{(0)} = m(\langle \ln \mu \rangle / \langle \ln s^{-1} \rangle)$. Physically T is the timescale of the ‘stirring force’ at level L . We once more recover the deterministic results of [16, 17] when $\hat{\sigma}_s^2, \hat{\sigma}_\mu^2$ vanish. Now observe, however, that the corrections to the deterministic case come in with a negative sign. On the other hand, for $t > T$, the non-trivial time dependence of $\langle\langle x^m \rangle\rangle_c(t)$ comes from the second sum in (3.22). The computation proceeds along the same lines as before, and we get, finally,

$$\langle\langle x^m \rangle\rangle_c(t) \sim t^{\theta_m^{(0)}} (t/T)^{\frac{1}{2} \text{sgn}[\ln(t/T)] m \theta_m^{(0)} (\ln \mu) (\hat{\sigma}_\mu^2 + \hat{\sigma}_s^2)}. \tag{3.26}$$

The behaviour of θ_m with m for $t \geq T$ has been sketched in figure 5. Strictly speaking there is a crossover region, where $\langle \ln s^{-1} \rangle |L - \bar{l}| \sim |\ln(t/T)|$ is small, and where the application of the CLT is not justified, so that equation (3.26) should really be interpreted as being valid for $t/T \gg 1$ or $t/T \ll 1$, with t and T both large.

The other quantity of interest that we can calculate is the exponent ν of the correlation decay and this we will do also taking the form (2.7) for the transition rates. The autocorrelation function, i.e. the probability of return to the origin at time t , having been there at $t = 0$, is given by

$$\mathcal{P}_0(t) = \sum_{i=1}^{\infty} \left\langle \prod_{i=1}^i z_i^{-1} q_0^{(i)}(t) \right\rangle_c \tag{3.27}$$

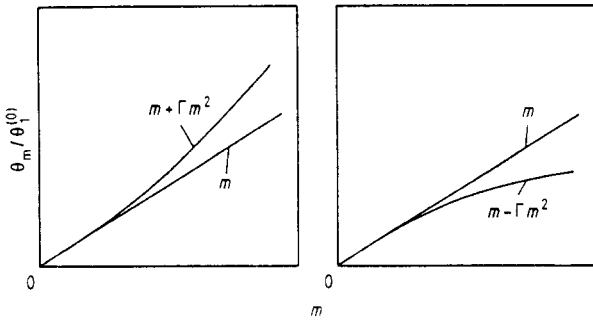


Figure 5. The exponent of the temporal moment growth θ_m against the order m of the moment, in region A. If there are no parameter fluctuations one has $\theta_m = m\theta_1^{(\sigma)}$ with $\theta_1^{(\sigma)}$ defined by $\ln \mu / \ln s^{-1}$. If the parameters fluctuate, $\theta_1^{(\sigma)} = \langle \ln \mu \rangle / \langle \ln s^{-1} \rangle$ and there is a quadratic correction to θ_m . Its magnitude Γ depends on the relative variances, $\Gamma = \frac{1}{2} \langle \ln \mu \rangle (\hat{\sigma}_z^2 + \hat{\sigma}_u^2)$, and its sign on whether the diffusion occurs below (-) or above (+) the scale setting level L . The expression is valid for small variance, $m\Gamma \ll 1$.

where z_i are the branching ratios at each level i , each $z_i > 1$, so that the sum is rapidly converging. Making a one-pole approximation to the function $q_0^{(l)}(t)$, where we just take the smallest eigenvalue, we find in region B where $\lambda_0^{(l)} = \prod_{i=1}^l u_i$, that

$$\mathcal{P}_0(t) \sim t^{-\nu} \tag{3.28}$$

up to logarithmic factors, with

$$\nu \approx \nu_u [1 - \frac{1}{2} \langle \ln z \rangle (\hat{\sigma}_z^2 + \hat{\sigma}_u^2)] \tag{3.29}$$

where $\nu_u \equiv \langle \ln z \rangle / \langle \ln u^{-1} \rangle$ and where we have once more made use of the CLT and kept terms up to first order in the (relative) variances. In this case there is no dependence at all on the level L at which the system is prepared.

In region A we find, once more taking a one-pole approximation to $q_0^{(l)}(t)$, i.e. keeping only the leading relaxation time,

$$\mathcal{P}_0(t) \sim t^{-\nu_s (1 - \frac{1}{2} \langle \ln z \rangle \hat{\sigma}_z^2)} (t/T)^{\frac{1}{2} \text{sgn}[\ln(t/T)] \nu_s \langle \ln z \rangle \hat{\sigma}_z^2} \tag{3.30}$$

where we have defined $\nu_s = \langle \ln z \rangle / \langle \ln s^{-1} \rangle$; T is defined as before. Note that in both cases we recover the ‘deterministic’ results of [16, 17] when the relative variances are set to zero. The oscillatory behaviour [10, 26] of $\mathcal{P}_0(t)$ is no longer evident, in the general case, due to the destruction of the self-similarity of the functions $q_0^{(l)}(t)$ under the scaling of t by a uniform u or s in the appropriate regions. However notice that this self-similarity is restored *on the average* in the case we have been treating in this section.

4. Discussion and application to turbulence

In the deterministic case—without fluctuations—a generalised hyperscaling relation is satisfied by θ_m , ν and the fractal dimensionality $D_F = \ln z / \ln \mu$, namely

$$\nu = D_F \theta_m / m \tag{4.1}$$

in those parameter regions where only one time-scaling parameter is relevant. This happens to be region A, where $\lambda_0^{(l)} \approx s^l$ and the hierarchy of relaxation times scales

with s , and regions B, C with $\mu^m rs < 1$ ($\lambda_0^{(l)} \sim (rs)^l \ll \lambda_1^{(l)}$, scale parameter rs). By contrast, in regions B, C taking $\mu^m rs > 1$ brings into play the cutoff $l_s \sim \ln t / \ln s^{-1}$ (cf [16, 17]); although the transitions are predominantly downward ($rs = W_k^+ / W_k^- < 1$) the overall mean displacement is limited by the rate of progress up the levels l , $\lambda_1^{(l)} \sim s^l$. The interplay between the two different self-similar sets of relaxation times with different scale parameters destroys the hyperscaling relation. Generalising the definition of D_F to the case where the spatial scale factor μ and the branching ratio z are randomly distributed from level to level, we may write, for any given configuration,

$$\lim_{n \rightarrow \infty} \prod_{i=1}^n z_i \mu_i^{-D_F} = 1$$

from which follows

$$D_F = \frac{\langle \ln z \rangle}{\langle \ln \mu \rangle} \tag{4.2}$$

Comparing (3.29) and (3.30) with (3.21) and (3.26) we immediately see that the terms proportional to the variances break the hyperscaling relation between ν and θ_m . This is to be expected in view of the above discussion. On the other hand the fact that for the particular choice for the W_k^\pm treated in § 3 self-similarity is preserved on the average, ensures power-law behaviour for the moments and the autocorrelation function.

We would now like to discuss the possible relevance of the results found in the previous section for diffusion on a random fractal to the case of diffusion in a turbulent medium. According to ‘classical’ scaling theories of fully developed turbulence [34], there is a detailed balance of energy transfer between eddies of different scales. This condition of exact detailed balance has been relaxed by various authors, who have considered, instead, detailed balance on the average as, for example, in the so-called log-normal model [35] or the random- β model recently introduced by Benzi *et al* [36]. The random model which we have introduced in this paper goes one step further than the random- β model in that it involves random fractal sets in both space [36, 37] and time [14, 38-40].

We define the ratio of the energy transfer per unit mass on the scale of level $l-1$ to the same quantity on the scale of level l to be

$$\epsilon_{l-1} / \epsilon_l = \mu_l^{-D_F} \mu_l^{-2} s_l^{-3} \tag{4.3}$$

The factor $\mu_l^{-2} s_l^{-3}$ arises from the fact that ϵ has the dimension of (length)² (time)⁻³, while the factor μ^{-D_F} has been included due to the fact that the eddies are not space filling [39] but are concentrated on a multifractal set of dimension $D_F < d$, d being the Euclidean dimension of the embedding space.

Requiring that

$$\lim_{N \rightarrow \infty} \left(\prod_{l=1}^N \mu_l^{-\Delta} s_l^{-3} \right)^{1/N} = \langle \mu^{-\Delta} s^{-3} \rangle_c = 1 \tag{4.4}$$

where we have defined $\Delta \equiv 2 + D_F - d < 2$, implies, once more assuming that the μ_l and s_l are independently distributed and using the CLT,

$$\Delta \langle \ln \mu \rangle + 3 \langle \ln s \rangle = \frac{1}{2} \Delta^2 \sigma_\mu^2 + \frac{9}{2} \sigma_s^2 \tag{4.5}$$

up to leading order in the variances. It is in region A that the exponent θ_m is larger than one (cf [16]), promising to be the region with which to model the turbulent medium. Solving for $\langle \ln s^{-1} \rangle$ in (4.5) and substituting in (3.26) we find

$$\langle \langle x^m \rangle \rangle_c(t) \sim t^{(3m/\Delta)[1 + \frac{1}{2} \Delta \langle \ln \mu \rangle (\sigma_\mu^2 + \sigma_s^2)]} (t/T)^{\frac{1}{2} \text{sgn}[\ln(t/T)] (3m^2/\Delta) \langle \ln \mu \rangle (\sigma_\mu^2 + \sigma_s^2)} \tag{4.6}$$

where again we have kept only the lowest-order terms in $\hat{\sigma}_v^2$ in the exponent. We see that with large m , the corrections in $\hat{\sigma}_v^2$ become negative for $t < T$, while for $t > T$ they are positive for all m .

These findings agree with those obtained for the velocity structure functions from the log-normal model [35] for $t < T$.

Contact can be made with the intermittency corrections calculated via the log-normal model [35] or the β -model [41, 36] by noting that $\langle x^m \rangle(t)$, which should be interpreted as the m th moment of the relative displacement of two diffusing particles, can be expressed in terms of the m th-order relative velocity correlation function as follows [42]:

$$\langle x^m \rangle(t) \sim \int_0^t \dots \int_0^t dt_1 \dots dt_m \langle v(t_1) \dots v(t_m) \rangle. \tag{4.7}$$

Claiming that $\langle v(r(t_1) \dots v(r(t_m))) \rangle \sim \langle v^m(r) \rangle$, with $\langle v^m(r) \rangle \sim r^{\zeta_m}$ (this r denotes the eddy size and should not be confused with our parameter r , cf (2.6b)), we have

$$\langle x^m \rangle(t) \sim r^{\zeta_m} t^m.$$

Taking for the average relative displacement $r^2 \sim t^{\theta_2}$ yields

$$\theta_m = \frac{1}{2} \zeta_m \theta_2 + m. \tag{4.8}$$

Thus ζ_m and $(\theta_m - m)$ are expected to show the same trend in m . We find this indeed to be the case at least for $m(\hat{\sigma}_\mu^2 + \hat{\sigma}_u^2) \ll 1$ comparing our results for $t < T$ (all length scales below the deterministic scale, i.e. the scale of the stirring force) with those of Benzi *et al* [34] where the stirring length is taken to be the largest length scale in the model. One should also compare (4.8) with our findings for the two-delta-function distribution, valid to all orders of the variances, in the region $t > T$. Recall that $\theta_m \sim m$ for large m in this case, while ζ_m , as given by Benzi *et al* [36] tends to a constant for large m .

In concluding this section we would like to compare our results with those previously found for RW on other fractal substrates. We note that one may identify the RW dimension [43], defined as $t \sim \langle x \rangle^{d_w}$, with our θ_1^{-1} , namely

$$d_w = \frac{\langle \ln \cdot \rangle}{\langle \ln \mu \rangle} \tag{4.9}$$

where the argument in the numerator is s^{-1} in region A and u^{-1} in region B with $\mu_k^m u_k < 1$, almost all k . Our model then affords a case where one may explicitly test conjectures relating d_w and D_F , the fractal dimension of the substrate where the RW takes place. In particular, in the above application to turbulence, neglecting fluctuations,

$$d_w = \frac{1}{3}(2 + D_F - d) \tag{4.10}$$

from (4.6). This is to be compared with the Aharony-Stauffer [44] conjecture

$$d_w = D_F + 1 \tag{4.11}$$

and the Alexander and Orbach rule [45] implied by $2\nu \equiv \frac{4}{3}$,

$$d_w = \frac{3}{2} D_F. \tag{4.12}$$

The relations (4.11) and (4.12) are inconsistent with (4.10) except at the particular values $D_F = -(d + 1)/2$ and $D_F = 2(2 - d)/7$ respectively, both clearly unphysical for $d = 3$.

5. Non-algebraic behaviour of moments and correlations

In this section we would like to give two further examples for the way in which the transition rates may be distributed, namely, cases (ii) and (iii) of § 1, yielding, respectively, a stretched exponential and an exponential-logarithmic form for the moments.

Case (ii). Choose $W_l^+ = (\sum_{i=1}^l v_i)^{-\gamma}$, $\gamma > 0$, $L = 0$. The v_i are independently distributed random variables, constrained to be > 0 for almost all i . This automatically ensures that $W_l^+ / W_{l-1}^+ < 1$, for almost all l . If we require, in addition, that $W_l^+ / W_{l+1}^+ > 1$ for almost all l , we find ourselves in the parameter region A (figure 3). From equation (3.3) and (3.4) and the fact that $g(\tau) = dG(\tau)/d\tau$ it follows that the m th moment of the displacement from the origin is now given by

$$\langle\langle x^m \rangle\rangle_c(t) \sim \sum_{l=0}^{\infty} e^{lU_m^+} \langle G_{l-1}(W_{l-1}^+ t) - G_l(W_l^+ t) \rangle_c. \tag{5.1}$$

This should be compared with (3.22) with $L = 0$. Making a one-pole approximation to the $G_l(\tau)$, we rearrange (5.1) to be

$$\langle\langle x^m \rangle\rangle_c(t) \sim [\exp(U_m^+) - 1] \sum_{l=0}^{\infty} \exp(lU_m^+) \langle 1 - \exp(-y^{-\gamma} t) \rangle_l \tag{5.2}$$

where the average is now to be taken with respect to the Gaussian distribution for $y = \sum_{i=1}^l v_i$

$$P_l(y) \sim \exp[-\frac{1}{2}(y - l\bar{v})^2 / l\sigma_v^2]$$

for l large enough, under the usual conditions for the v distribution; $\bar{v} = \langle v \rangle$, $\sigma_v^2 = \langle (\delta v)^2 \rangle$. Performing a saddle-point approximation to the Gaussian average as well as to the sum over l gives, at the saddle point, $\bar{l} \propto t^{1/\gamma}$. Expanding the exponent to first order in the relative variances, we have

$$\langle\langle x^m \rangle\rangle_c(t) \sim \exp\{ (m \langle \ln \mu \rangle / \bar{v}) [1 + \frac{1}{2} m \langle \ln \mu \rangle (\hat{\sigma}_\mu^2 + \hat{\sigma}_v^2)] t^{1/\gamma} \}. \tag{5.3}$$

The calculation of the autocorrelation function proceeds along the same lines as before. Substituting $q_0^{(l)}(t) \sim e^{-ly^{-\gamma}}$ in (3.27) and once more making saddle-point approximations to the integrals over l and y , with the weight $P_l(y)$ as given above, we find

$$\mathcal{P}_0(t) \sim \exp(-Ct^{1/(\gamma+1)}). \tag{5.4}$$

The constant coefficient in the exponent is

$$C = (\gamma + 1) \left(\frac{\langle \ln z \rangle}{\gamma \bar{v}} \left[1 - \frac{1}{2} \langle \ln z \rangle (\hat{\sigma}_z^2 + \hat{\sigma}_v^2) \right] \right)^{\gamma/(\gamma+1)}.$$

Hence the moments grow and the correlations decay according to a law of stretched exponentials, but with different exponents, the correlations decreasing slower. The order m of the moments only affects the prefactor of the stretched exponential but not the t exponent, $1/\gamma$, that is only related to the W_l^+ scaling.

Our results should be compared with those of De Dominicis *et al* [11, 25, 27]. Algebraic scaling of the spectrum with l leads to stretched exponentials, while exponential scaling of the spectrum implies algebraic time dependence.

Case (iii). Next we consider the choice $W_k^+ = w_k^k$, $W_k^- = w_k^{k-1}/r$, where the w_k are identically distributed variables living on the level k . If the w_k have a small variance, the overall behaviour of the functions $g_t(\tau)$ and $G_t(\tau)$ is not altered. For $w_k < 1$, $W_k^+/W_{k+1}^- > 1$, almost all k , $\langle\langle x^m \rangle\rangle_c(t)$ is once more given by (5.1), and we may write, analogously to (3.23),

$$\langle\langle x^m \rangle\rangle_c(t) \sim \int dl \int dw_l e^{U_m^+ l} \delta(l - \ln t / \ln w_l^{-1}) p(w_l) \tag{5.5}$$

where $p(w)$ is the probability distribution for w . Note that performing the l integral gives rise to the factor $\exp(U_m^+ \ln t / \ln w^{-1})$ in the integrand, and this diverges exponentially as $w \rightarrow 1$. If $p(w) = 0$ for $w > w_>$ ($w_>$ is some constant less than 1), centred at some finite value \bar{w} and having a small relative variance, one obtains an algebraic growth law with logarithmic corrections.

We now consider a distribution that decays sufficiently fast as $w \rightarrow 1$ so that the w integral in (5.5) still converges, namely

$$p(w) = \begin{cases} 0 & w < w_< & w_< > 0 \\ \text{constant} \times \exp[-(1-w)^{-\alpha}] & w \rightarrow 1 & \alpha > 1 \end{cases} \tag{5.6}$$

and arbitrary otherwise. Making a saddle-point approximation to the w integral in (5.5), we see that for t sufficiently large ($(U_m^+ \ln t)^{1/\alpha} \gg 1$), the result does not depend on $w_<$. We find

$$\langle\langle x^m \rangle\rangle_c(t) \sim \exp[\alpha^{\alpha/(1-\alpha)} (\alpha - 1) (U_m^+ \ln t)^{\alpha/(\alpha-1)}]. \tag{5.7}$$

It can be seen that the exponent blows up for $\alpha \rightarrow 1$, signalling an exponential behaviour. The larger α is, the larger t has to be for the asymptotic behaviour to be observed. Note that for $\alpha \gg 1$ and sufficiently large t the behaviour of $\langle\langle x^m \rangle\rangle_c(t)$ will be indistinguishable from a power law, with the effective exponent $\theta_m^{\text{eff}} = U_m^+$.

We have considered relaxation in three different families of systems, each of which possesses infinite, random hierarchies of relaxation times, with, however, different types of randomness introduced into each of them.

In case (i), the relaxation times form a statistically self-similar hierarchy with multiplicative disorder, obeying an approximately log-normal distribution. We find that the moments and correlation functions retain, up to logarithmic factors, the algebraic behaviour found in the homogeneous case, while the exponents acquire corrections due to disorder. It is interesting to note that the boundary conditions (the time and length scales at which the system is prepared, say, stirred) break the temporal and spatial scale invariance of the system and show up in the correlation functions with precisely an exponent which goes to zero with the relative variance of the random parameters (cf (3.26), (3.30)). The generalised hyperscaling relation between the moment (θ_m) and autocorrelation (ν) exponents, as well as the linear dependence of θ_m on the order of the moment, break down due to the disorder. In contrast, the hierarchy of transition rates in case (ii) constructed from an additive random process, namely, $(W_k^+)^{-1/\gamma} = \sum_i^k v_i$ where the v_i are independently distributed random variables, leads to moments and correlation functions exhibiting a stretched exponential behaviour with the same exponent as found for the pure case, although different coefficients. The third model (case (iii)) reduces, along with case (i), to the homogeneous hierarchical model considered by Grossman *et al* [16] in the limit of vanishing disorder; however, for non-zero variances it gives rise to a novel exponential-logarithmic behaviour for the moments, which has been found before in the case of hopping conduction [31].

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References

- [1] Blumen A, Klafter J and Zumofen G 1986 *Optical Spectroscopy of Glasses* ed I Zschokke-Granacher (Dordrecht: Riedel) p 199
- [2] Gefen Y, Aharony A and Alexander S 1983 *Phys. Rev. Lett.* **50** 77
- [3] Rammal R and Toulouse G 1983 *J. Physique Lett.* **44** L13
- [4] Bunde A 1986 *Advances in Solid State Physics* vol 26, ed P Grosse (Braunschweig: Vieweg) p 113
Mitescu C D and Rousseng J 1983 *Percolation Structures and Processes* (*Ann. Israel Phys. Soc.*) vol 5, ed G Deutscher, R Zallen and J Adler (Bristol: Adam Hilger) p 81
- [5] Mézard M, Parisi G, Sourlas N, Toulouse G and Virasoro M 1984 *Phys. Rev. Lett.* **52** 1156; 1984 *J. Physique* **45** 843
- [6] Rammal R, Toulouse G and Virasoro M A 1986 *Rev. Mod. Phys.* **58** 765
- [7] Palmer R G, Stein D L, Abrahams E and Anderson P W 1984 *Phys. Rev. Lett.* **53** 958
Ogielsky T and Stein D L 1985 *Phys. Rev. Lett.* **55** 1634
- [8] Huberman B A and Kerszberg M 1985 *J. Phys. A: Math. Gen.* **18** L331
- [9] Teitel S and Domany E 1985 *Phys. Rev. Lett.* **55** 2176; 1986 *Phys. Rev. Lett.* **56** 1755
- [10] Schreckenberg M 1985 *Z. Phys. B* **60** 483
- [11] Paladin G, Mézard M and De Dominicis C 1985 *J. Physique Lett.* **46** L985
- [12] Maritan A and Stella A 1986 *J. Phys. A: Math. Gen.* **19** L269; 1986 *Phys. Rev. Lett.* **56** 1754; 1986 *Phys. Rev. B* **34** 456
- [13] Sibani P 1986 *Phys. Rev. B* **34** 3555
- [14] Blumen A, Klafter J and Zumofen G 1986 *J. Phys. A: Math. Gen.* **19** L77
- [15] Wegner F and Grossmann S 1985 *Z. Phys. B* **59** 197
- [16] Grossmann S, Wegner F and Hoffmann K H 1985 *J. Physique Lett.* **46** L575
- [17] Hoffmann K H, Grossmann S and Wegner F 1985 *Z. Phys. B* **60** 401
- [18] Honerkamp J and Baumgärtner A 1986 *Z. Phys. B* **64** 253
- [19] Kumar D and Shenoy S R 1986 *Solid State Commun.* **57** 927
- [20] Derrida B 1983 *J. Stat. Phys.* **31** 433
- [21] Alexander S, Bernasconi J, Schneider W R and Orbach R 1981 *Rev. Mod. Phys.* **53** 175
- [22] Bernasconi J, Schneider W R and Wyss W 1980 *Z. Phys. B* **37** 175
- [23] Bernasconi J and Schneider W R 1985 *Helv. Phys. Acta* **58** 597
- [24] Rammal R 1985 *J. Physique* **46** 1837
- [25] De Dominicis C, Orland H and Lainee F 1985 *J. Physique Lett.* **46** L463
- [26] Schreckenberg M 1985 *PhD Thesis* Köln
- [27] De Dominicis C 1986 *J. Magn. Magnet. Mater.* **54-57** 17
- [28] Bachas C P and Huberman B A 1986 *Phys. Rev. Lett.* **57** 1965
- [29] Derrida B 1980 *Phys. Rev. Lett.* **45** 79; 1981 *Phys. Rev. B* **24** 2613
- [30] Blumen A, Klafter J and Zumofen G 1986 *Fractals in Physics* ed L Pietronero and E Tosatti (Amsterdam: Elsevier) p 399
- [31] Scher H and Lax M 1973 *Phys. Rev. B* **7** 4491, 4502
- [32] Mandelbrot B B 1974 *J. Fluid Mech.* **62** 331
- [33] Hoffmann K H 1986 private communication
- [34] Monin A S and Yaglom A M 1975 *Statistical Fluid Mechanics* (Cambridge, MA: MIT Press)
- [35] Oboukov A M 1962 *J. Fluid Mech.* **13** 77
Kolmogorov A A 1962 *J. Fluid Mech.* **13** 82
Yaglom A M 1966 *Sov. Phys.-Dokl.* **2** 26
- [36] Benzi R, Paladin G, Parisi G and Vulpiani A 1984 *J. Phys. A: Math. Gen.* **17** 3521
- [37] Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B I 1986 *Phys. Rev. A* **33** 1141

- [38] Blumen A, Klafter J, White B S and Zumofen G 1984 *Phys. Rev. Lett.* **53** 1301
- [39] Bendler J T and Schlesinger M F 1984 *Relaxations in Complex Systems* ed K L Ngai (Washington DC: Naval Research Laboratory) p 261
- [40] Montroll E W and West B J 1979 *Fluctuation Phenomena* ed E W Montroll and J L Lebowitz (Amsterdam: North Holland) p 61
- [41] Frisch U, Sulem P-L and Nelkin M 1978 *J. Fluid Mech.* **87** 719
- [42] Grossmann S and Procaccia I 1984 *Phys. Rev. A* **29** 1358
Effinger H and Grossman S 1984 *Phys. Rev. Lett.* **53** 442
- [43] Gefen Y, Aharony A and Alexander S 1983 *Phys. Rev. Lett.* **50** 77
- [44] Aharony A and Stauffer D 1984 *Phys. Rev. Lett.* **52** 2368
- [45] Alexander S and Orbach R 1982 *J. Physique Lett.* **43** L625